$I(G) = \{T_a : a \in G\}$

3.1.5. Homomorphism of Abelian Groups

If G and H are abelian (i.e., commutative) groups, then the set Hom(G, H) of all group homomorphisms from G to H is itself an abelian group. The sum (h + k) of two homomorphisms is defined by,

$$(h + k)(u) = h(u) + k(u) \quad \forall u \in G.$$

To verify that (h + k) is again a group homomorphism the commutativity of H is required. The sum of homomorphisms is favorable with the structure of homomorphism. It means that, if f is in Hom(K, G), h, k are elements of Hom(G, H), and g is in Hom(H, L), then,

$(h + k) \circ f = (h \circ f) + (k \circ f)$ and
$g \circ (h + k) = (g \circ h) + (g \circ k)$.

Example 1: Suppose A = {0, 1} and the semigroups (A*, •) and (A, +)
Where • is used to denote the concatenation operation and + is defined by the table given below:

+	0	1
0	0	1
1	1	0

The function f: $A^* \rightarrow A$ is defined by

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ has an odd number of 1's} \\ 0 & \text{if } \alpha \text{ has an even number of 1's.} \end{cases}$$

Show that f is homomorphic.

Solution: It is clear from table that if α and β are any elements of A^*, then

$$f(\alpha \cdot \beta) = f(\alpha) + f(\beta).$$

Here, f is a homomorphism. The function f is onto since,
$f(0) = 0;\ f(1) = 1$
It is not one to one.

Example 2: When R and R_0 are the group of all real numbers under addition and group of all real numbers under addition respectively and $f : R \rightarrow R_0$ be defined by, $f(x) = 2^x,\ \forall x \in R$ then show that it is a homomorphism.

Solution: It is a homomorphism since $\forall$ x, y $\in$ R,

$$f(x + y) = 2^{x+y} = 2^x \cdot 2^y = f(x)\, f(y)$$

Thus, $f : R \rightarrow R_0$ is a homomorphism.

Example 3: Prove that ϕ is a homomorphism when G is the group of integers under addition and $G' = G$, and let $\phi : G \to G'$ be defined by $\phi(x) = 2x$.

Solution: We know that, $\forall\ x, y \in G$
$\phi(x + y) = 2(x + y) = 2x + 2y = \phi(x) + \phi(y)$

Hence, $\phi\ G \to G'$ is a homomorphism.

Example 4: Show that the mapping $\phi : G'_+ \to G$ when G be the additive group of real numbers and G'_+ the multiplicative group of positive real numbers. This function is defined by $\phi(x) = \log_{10} x$ is a homomorphism.

Solution: We know that, $\forall\ x, y \in G'_+$

$$\phi(x\,y) = \log_{10}(x\,y) = \log_{10} x + \log_{10} y$$
$$= \phi(x) + \phi(y)$$

Hence, $\phi : G'_+ \to G$ is a homomorphism.

Example 5: Consider a group G and mapping $f : G \to G$ then prove that G is abelian when $f(x) = x^2$, $x \in G$ is a homomorphism.

Solution: Assume that an abelian group is G and $(xy)^2 = x^2y^2\ \forall\ x, y \in G$.

Now, we will show that:
$f : G \to G$ defined as $f(x) = x^2$ is a homomorphism

Let $x, y \in G \Rightarrow xy \in G$,
we have, $f(xy) = (xy)^2 = x^2y^2 = f(x)\,f(y)$

Hence, f is a homomorphism.

On the contrary, If a homomorphism is f from G to G then
$f(xy) = f(x)\,f(y)\ \forall\ x, y \in G$
$\Rightarrow (xy)^2 = x^2y^2 \Rightarrow xy\,xy = xx\,yy$

From the cancellation law, we have
$\Rightarrow yx = xy\ \forall\ x, y \in G$

Thus, G is abelian. Hence proved.

Example 6: Consider a group G and mapping $f : G \to G$ then prove that G is abelian when $f(x) = x^{-1}$, $x \in G$, is a homomorphism.

Solution: Assume that G is an abelian group and $f : G \to G$ such that

$$f(x) = x^{-1}\ \forall\ x \in G$$

Let us show that f is a homomorphism.

Let $x, y \in G$ be arbitrary elements of G then

$$f(xy) = (xy)^{-1} = y^{-1}\,x^{-1}$$

by reversal law and G is abelian
$x^{-1}\,y^{-1} = f(x)\,f(y)$
Thus f is a homomorphism.

Conversely, let $f : G \to G$ defined by $f(x) = x^{-1}$, for all $x \in G$ be homomorphism.

We have to show that G is abelian

Let $x, y \in G$ be arbitrary elements of G then,
$f(xy) = f(x)\,f(y)$